From Math to Hardware

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Introduction

- CPUs hardly any development
- FPGAs on the rise
- CλaSH: Haskell $\Rightarrow$ VHDL/Verilog

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  They are big, world wide companies (search engines, networking, high frequency trading, car automation, ...)

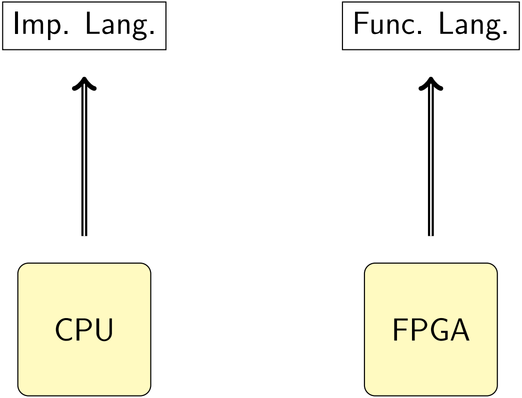
Introduction

QBAYLOGIC

- Ripple Carry adder
- Processor
- PID controller
- Transformations

- Examples: Cochlea Membrane, Adaptive Cruise Control, Slam algorithm, *N*-Queens, Processors, PID controllers, Tunnelling Ball Device
- Elementary architectures: adders, multipliers

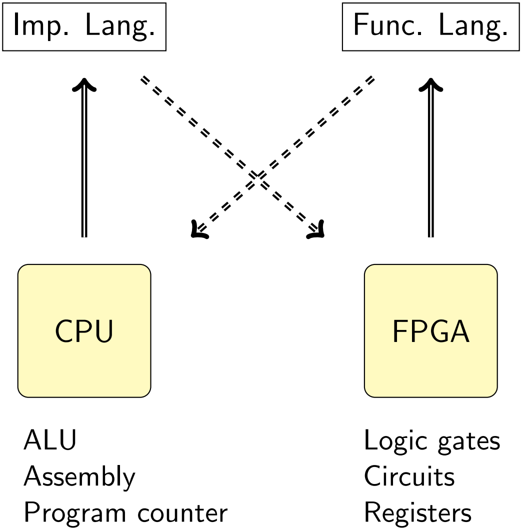
Languages and architectures



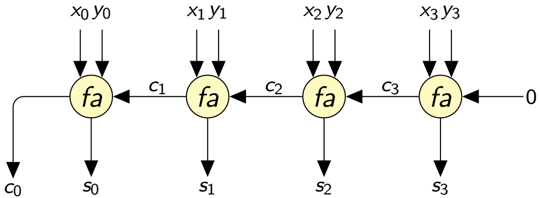
ALU
Assembly
Program counter

Logic gates
Circuits
Registers

Languages and architectures



Ripple Carry Adder



Logic Gates

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

$\otimes$	0	1
0	0	1
1	1	0

Logic Gates

$\wedge$	0	1
0	0	0
1	0	1

$$\begin{aligned} 0 \wedge 0 &= 0 \\ 0 \wedge 1 &= 0 \\ 1 \wedge 0 &= 0 \\ 1 \wedge 1 &= 1 \end{aligned}$$

$\vee$	0	1
0	0	1
1	1	1

$$\begin{aligned} 0 \vee 0 &= 0 \\ 0 \vee 1 &= 1 \\ 1 \vee 0 &= 1 \\ 1 \vee 1 &= 1 \end{aligned}$$

$\otimes$	0	1
0	0	1
1	1	0

$$\begin{aligned} 0 \otimes 0 &= 0 \\ 0 \otimes 1 &= 1 \\ 1 \otimes 0 &= 1 \\ 1 \otimes 1 &= 0 \end{aligned}$$

Half Adder

x	y	(c, s)
0	0	$(0, 0)$
0	1	$(0, 1)$
1	0	$(0, 1)$
1	1	$(1, 0)$

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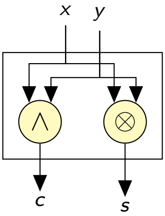
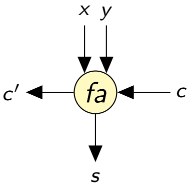


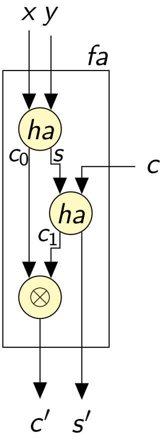
$$
\begin{aligned}
ha\ (x, y) \quad &= \quad (c, s) \\
&\textbf{where} \\
&\quad c \ = x \wedge y \\
&\quad s \ = x \otimes y
\end{aligned}
$$

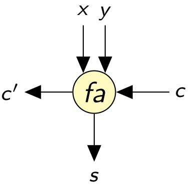
Full adder



x	y	c	(c', s)
0	0	0	$(0, 0)$
0	0	1	$(0, 1)$
0	1	0	$(0, 1)$
0	1	1	$(1, 0)$
1	0	0	$(0, 1)$
1	0	1	$(1, 0)$
1	1	0	$(1, 0)$
1	1	1	$(1, 1)$

Full adder





x	y	c	(c', s)
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1	0	0	$(0,1)$
1	0	1	$(1,0)$
1	1	0	$(1,0)$
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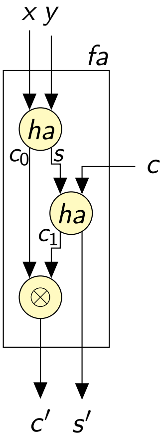
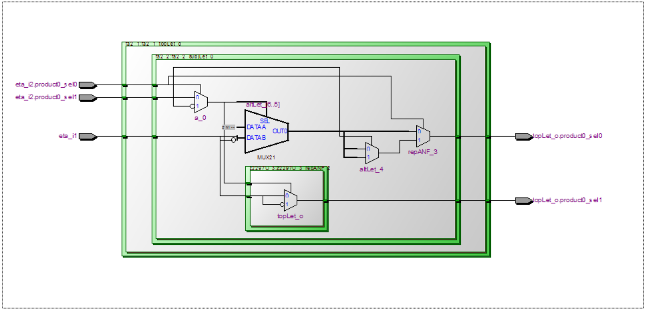
$$
\begin{aligned}
fa\ c\ (x, y) &= (c', s') \\
&\textbf{where} \\
&\quad (c_0, s) = ha\ (x, y) \\
&\quad (c_1, s') = ha\ (s, c) \\
&\quad c' = c_0 \otimes c_1
\end{aligned}
$$

Full adder

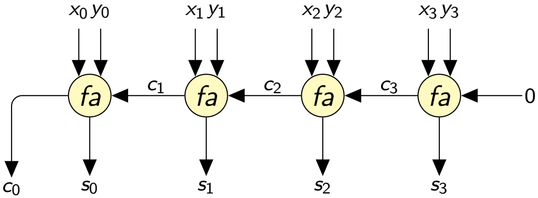


x y
fa
ha
c_0
s
c
ha
c_1
c′ s′



eta_i2.product0_sel0
eta_i2.product0_sel1
eta_i1
a_0
altLet_6
SEL
DATAA
DATAB
OUT0
MUX21
repANF_3
altLet_4
topLet_o
topLet_o.product0_sel0
topLet_o.product0_sel1

Ripple Carry Adder



$$fa\ c\ (x, y) \quad = \quad (c', s')$$

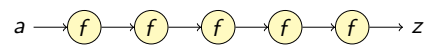
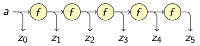
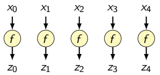
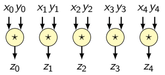
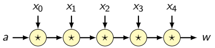
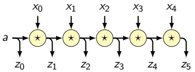
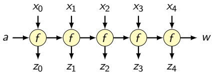
where

$\cdots$

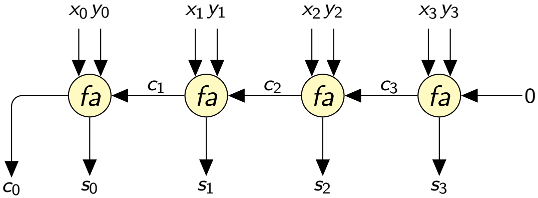
$$rca\ xs\ ys \quad = \quad \cdots$$

where

$\dots$

itn		$f\ x \Rightarrow z$	$z = f^n\ x$	$z = itn\ f\ x\ n$
itnscan		$f\ x \Rightarrow z$	$zs = \overline{f^n}\ x$	$zs = itnscan\ f\ x\ n$
map		$f\ x \Rightarrow z$	$zs = \widehat{f}\ xs$	$zs = map\ f\ xs$
zipWith		$x\ \star\ y \Rightarrow z$	$zs = xs\ \widehat{\star}\ ys$	$zs = zipWith\ (\star)\ xs\ ys$
foldl		$a\ \star\ x \Rightarrow a'$	$w = a \circledast xs$	$w = foldl\ (\star)\ a\ xs$
scanl		$a\ \star\ x \Rightarrow a'$ $z = a$	$w = a\ \overline{\circledast\ xs}$	$zs = scanl\ (\star)\ a\ xs$
mapAccumL		$f\ a\ x \Rightarrow (a', z)$	?	$(w, zs) = mapAccumL\ f\ a\ xs$

Ripple Carry Adder



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where

$\cdots$

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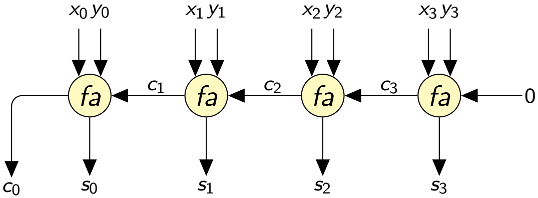


$$fa\ c\ (x, y) \;=\; (c', s')$$

where

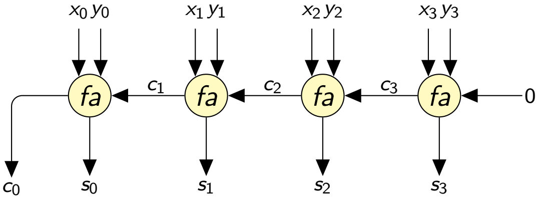
...

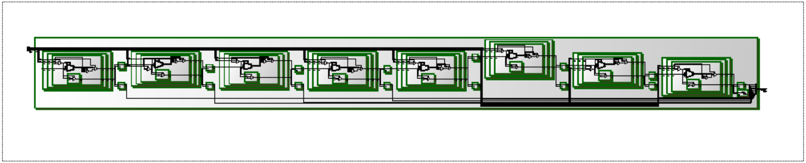
$$rca\ xs\ ys \;=\; c_0 : ss$$

where

$$xys \;=\; zip\ xs\ ys$$
$$(c_0, ss) \;=\; mapAccumR\ fa\ 0\ xys$$

Ripple Carry Adder





HPC Example: Heat Diffusion

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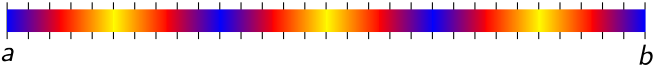
Problem:

Calculate temperature $T(t,x)$ for all time $t \geq 0$ and for each position $x \in [a,b]$.



Simplifications:

- One dimensional
- Temperatures of endpoints constant
- No external influences
- Endpoints constant temperature

Heatflow: Mathematics $\Rightarrow$ Discretization

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Specification:

$$\frac{\partial T(t,x)}{\partial t} - \alpha \cdot \nabla^2 T(t,x) = 0$$

($\partial t, \nabla$: diff. operators w.r.t. time, space)

Definitions:

$$\frac{\partial T(t,x)}{\partial t} = \lim_{\Delta t \to 0} \frac{T(t+\Delta t,x) - T(t,x)}{\Delta t}$$

$$\nabla^2 T(t,x) = \lim_{\Delta x \to 0} \frac{T(t,x-\Delta x) - 2T(t,x) + T(t,x+\Delta x)}{(\Delta x)^2}$$

Take Δt, Δx small, $\Rightarrow$ $t_k = k \cdot \Delta t$, $x_i = a + i \cdot \Delta x$:

$$\frac{T(t_{k+1},x_i) - T(t_k,x_i)}{\Delta t} = \alpha \cdot \frac{T(t_k,x_{i-1}) - 2T(t_k,x_i) + T(t_k,x_{i+1})}{(\Delta x)^2}$$

Write τ_i, τ_i' for $T(t_k, x_i), T(t_{k+1}, x_i)$, define $c = \frac{\alpha \cdot \Delta t}{(\Delta x)^2}$:

$$\tau_i' = \tau_i + c \cdot (\tau_{i-1} - 2\tau_i + \tau_{i+1})$$

Aggregation

$$\tau_i' = \begin{cases} \tau_0 & (i = 0) \\ \varphi\left(\tau_{i-1}, \tau_i, \tau_{i+1}\right) & (1 \leq i \leq N{-}1) \\ \tau_N & (i = N) \end{cases}$$

$$\mathit{next}\ \mathit{ts}\ =\ [\mathit{ts}!!0]\ +\!+\ \quad \mathit{map}\ \varphi\ (\mathit{triples}\ \mathit{ts})\ \quad +\!+\ [\mathit{ts}!!n]$$

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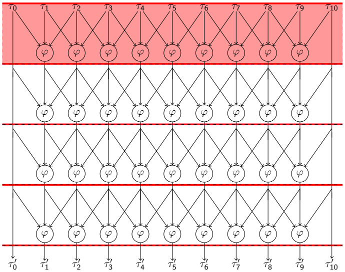

Aggregation

$$\tau'_i = \begin{cases} \tau_0 & (i = 0) \\ \varphi\,(\tau_{i-1}, \tau_i, \tau_{i+1}) & (1 \leq i \leq N{-}1) \\ \tau_N & (i = N) \end{cases}$$

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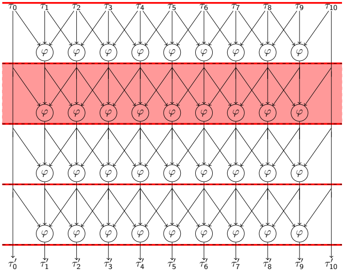

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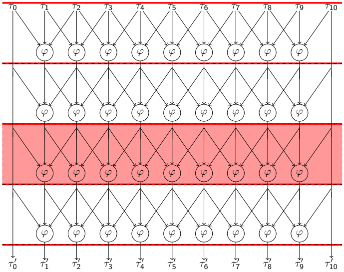

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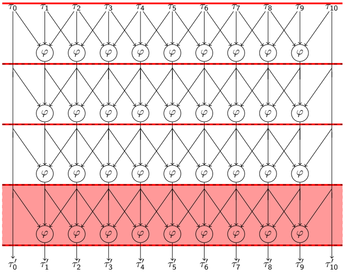
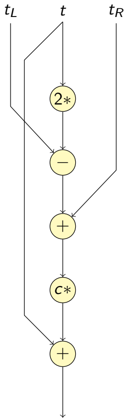
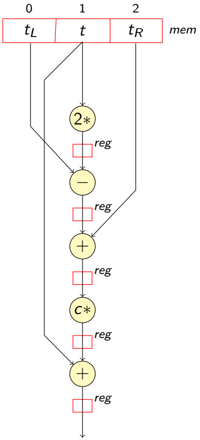


Processor derivation




```
f t (tL,tR) = t + c * (tL -2*t + tR)
```

Processor derivation

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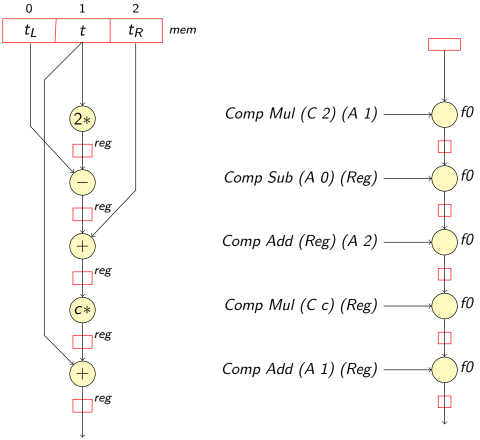


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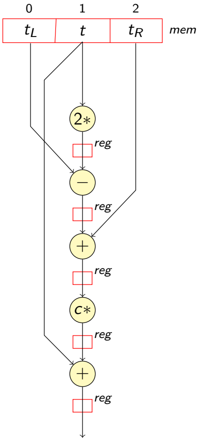

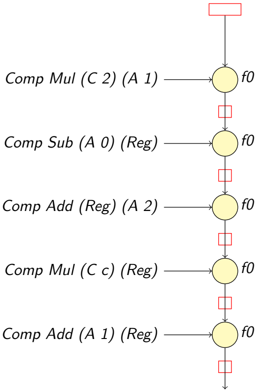


```haskell
data Data  = Addr Int
           | Const Int
           | Reg

data Opc   = Add
           | Mul
           | Sub
           | Nop

data Instr = Comp Opc Data Data
           | Read
```

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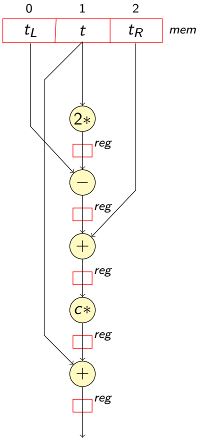

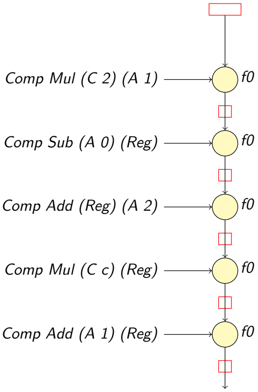


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prog0 = [ Comp Mul (Const 2) (Addr 1)
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        , Comp Add  Reg      (Addr 2)
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        , Comp Add (Addr 1)   Reg
        ]
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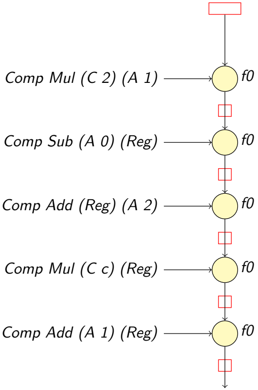


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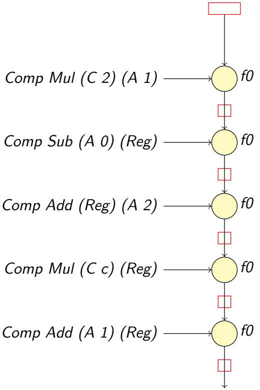


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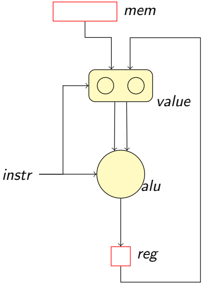
        , Comp Add (Addr 1)   Reg
        ]
```

```haskell
f0 mem reg (Comp opc d0 d1) = reg'
    where
      x = value (mem,reg) d0
      y = value (mem,reg) d1

      reg' = alu opc x y
```

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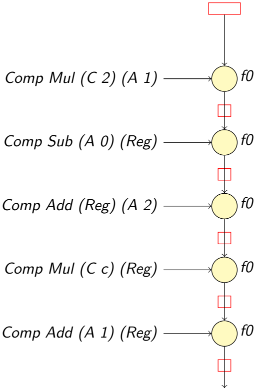


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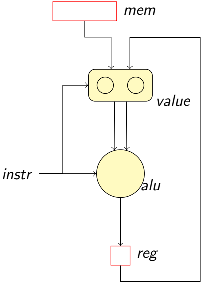


```
value (mem,reg) d = case d of
        Addr  i -> mem!!i
        Const n -> n
        Reg     -> reg

alu opc x y = case opc of
        Add     -> x + y
        Mul     -> x * y
        Sub     -> x - y
        Nop     -> x
```

```
f t (tL,tR) = t + c * (tL -2*t + tR)
```

```
prog = [ Read
       , Comp Mul (Const 2) (Addr 1)
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       , Comp Add  Reg      (Addr 2)
       , Comp Mul (Const c)  Reg
       , Comp Add (Addr 1)   Reg
       ]
```
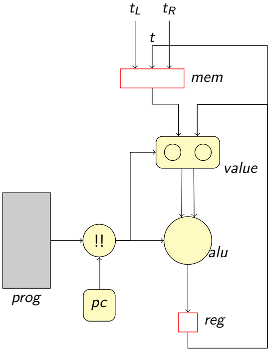
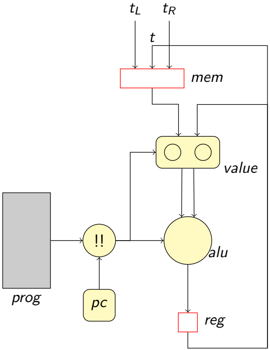



```haskell
prog = [ Read
       , Comp Mul (Const 2) (Addr 1)
       , Comp Sub (Addr 0)   Reg
       , Comp Add  Reg      (Addr 2)
       , Comp Mul (Const c)  Reg
       , Comp Add (Addr 1)   Reg
       ]
```

```haskell
f1 (mem,reg) (instr,(tL,tR)) = (mem',reg')
       where
         (mem',reg') = case instr of

                         Comp opc d0 d1  -> ( mem, alu opc x y )
                                            where
                                              x = value (mem,reg) d0
                                              y = value (mem,reg) d1

                         Read            -> ( [tL,reg,tR] , 0 )
```



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```
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                                       where
                                         x = value (mem,reg) d0
                                         y = value (mem,reg) d1

                    Read            -> ( [tL,reg,tR] , 0 )
```

```
f_proc prog (mem,reg,pc) (tL,tR) = ( (mem',reg',pc') , outp )

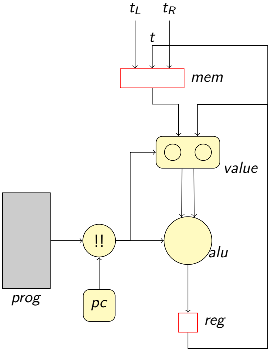
       where

         pc'         | pc < length prog -1     = pc + 1
                     | otherwise               = 0

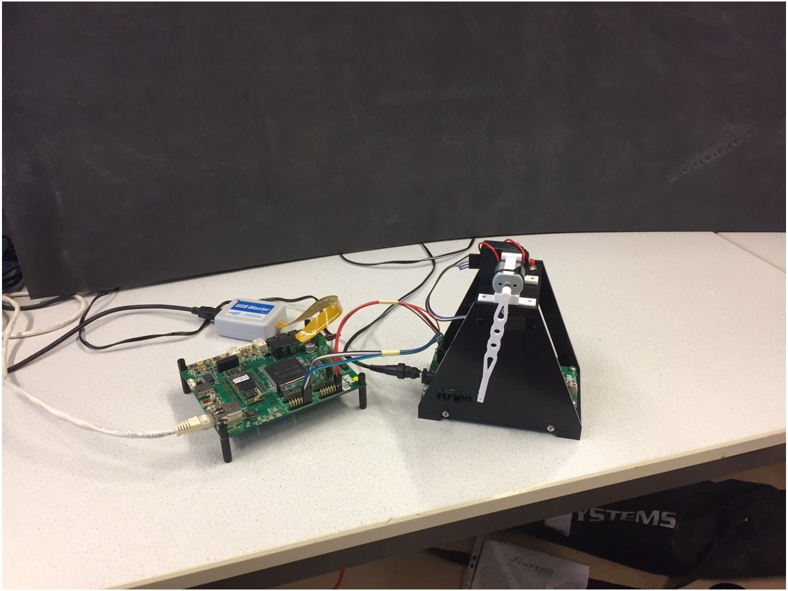
         instr       = prog!!pc

         (mem',reg') = f1 (mem,reg) (instr,(tL,tR))

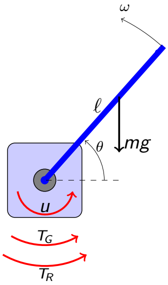
         outp        = reg
```



Pendulum

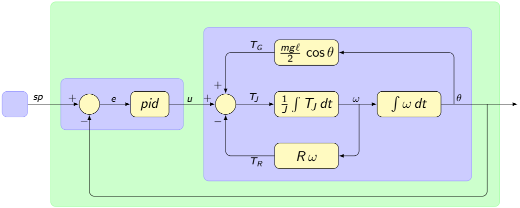


Pendulum

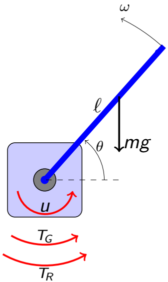


setpoints:
$45^\circ, 135^\circ$

Pendulum

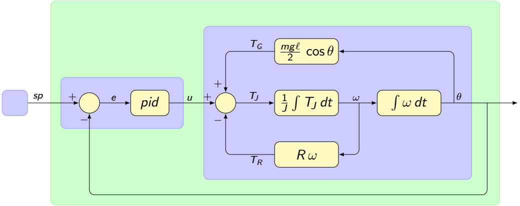


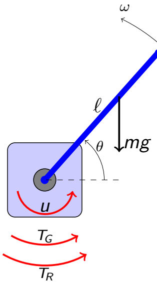
$$
\begin{aligned}
T_J(t) &= T_G + u - T_R \\
&\quad \textbf{where} \\
&\qquad T_G = \frac{mg\ell}{2}\cos(\theta(t)) \\
&\qquad u = pid(t) \\
&\qquad T_R = R\,\omega(t) \\
\\
\omega(t) &= \frac{1}{J}\int_0^t T_J(t)\,dt \\
\theta(t) &= \int_0^t \omega(t)dt
\end{aligned}
$$



setpoints:
$45^\circ, 135^\circ$

Pendulum





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\theta(t) &= \int_0^t \omega(t)dt
\end{aligned}
$$

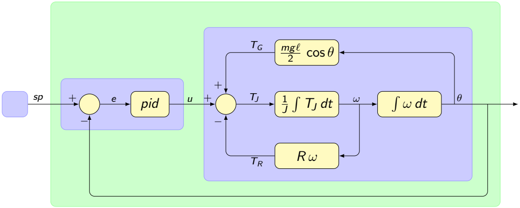
```
trq_J t = trq_G + u + trq_R
        where
          trq_G = m*g*l/2 * cos (theta t)
          u     = pid t
          trq_R = res * omega t

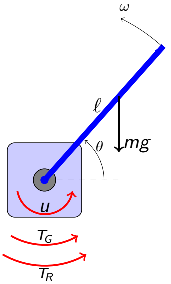
omega t = 1/j * integral trq_J (0,t-dt) dt

theta t = integral omega (0,t) dt
```

```
integral f (a,b) dx | b<=a      = 0
                    | otherwise = sum [ f x * dx | x <- [a, a+dx .. b-0.5*dx] ]
```

Pendulum





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omega t = 1/j * integral trq_J (0,t-dt) dt

theta t = integral omega (0,t) dt
```

$$
\begin{aligned}
e(t) &= sp(t) - \theta(t) \\
pid(t) &= K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}
\end{aligned}
$$

```
err t = sp t - theta t

pid t =   kp * err t
        + ki * integral err (0,t) dt
        + kd * (err t - err (t-dt)) / dt
```

Pendulum

```
trq_J t = trq_G + u + trq_R
        where
          trq_G = m*g*l/2 * cos (theta t)
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theta t = integral omega (0,t) dt
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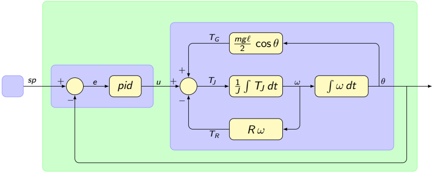

Pendulum

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      where
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        + ki * integral err (0,t) dt
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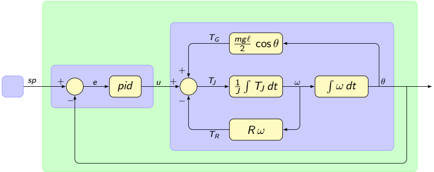
```




```
drivPend (sOmega,sTheta) u = ( (sOmega',sTheta') , th )
      where
        trq  = tG + u - tR
          where
            tG = m*g*l/2 * cos th
            tR = res * om

        (sOmega',om) = omega sOmega trq
        (sTheta',th) = theta sTheta om

omega int trq = ( int' , int )
      where
        int' = int + 1/j * trq*dt

theta int om  = ( int' , int' )
      where
        int' = int + om*dt
```

```
pid (int,prevE) (sp,mVal) = ((int',err), u)
      where
        err   = sp - mVal

        int'  = int + err*dt
        deriv = (err-prevE) / dt

        u     =   kp * err
                + ki * int'
                + kd * deriv
```

Pendulum

```
system (cyber,phys) (sCyber,sPhys) sp = ( (sCyber',sPhys') , mVal )
      where
        (sPhys' , mVal) = phys   sPhys   u
        (sCyber', u    ) = cyber sCyber (sp,mVal)
```

```
sim (system (pid,drivPend)) (sCyber_0,sPhys_0) setpoints
   where
     sCyber_0 = (0,0)
     sPhys_0  = (0,0)
     setpoints = ...
```

Towards CλaSH

- Number types:

  type *Int16* = *Signed 16*
  type *Nat32* = *Unsigned 32*
  type *Fix4_16* = *SFixed 4 16*

- Lists/Vectors:

  type *Prog* = *Vec 6 Instr*
  type *Mem* = *Vec 32 Int16*

- Top level:

  topEntity = f_proc prog <^> (mem0,0,0)

- To HDL:

  `:vhdl`
  `:verilog`

```
omega :: Float -> Float

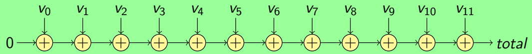
omega :: Float -> Float -> (Float,Float)

omega :: SFixed 18 18 -> SFixed 18 18 -> (SFixed 18 18, SFixed 18 18)
```

-

Transformation rules

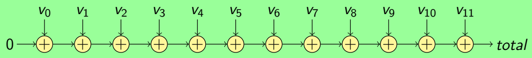
Transformation rules



$$total = foldl\ (+)\ 0\ vs$$

$$total = 0 \bigoplus vs$$

Transformation rules

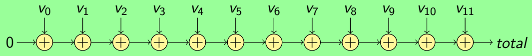


$$total = foldl\ (+)\ 0\ vs$$

$$total = 0 \oplus vs$$

$$\begin{aligned} foldl\ f\ a\ [\,] &= a \\ foldl\ f\ a\ (x{:}xs) &= foldl\ f\ (f\ a\ x)\ xs \end{aligned}$$

Transformation rules



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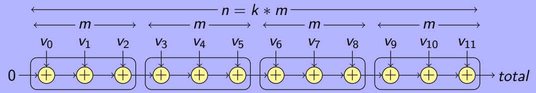

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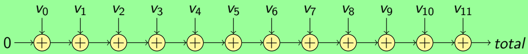

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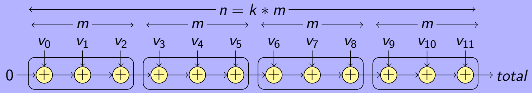
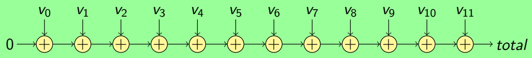


$$total = foldl\ (foldl\ (+))\ 0\ vss$$

Transformation rules



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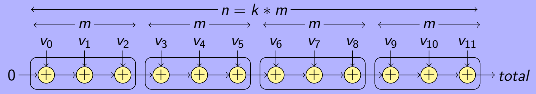

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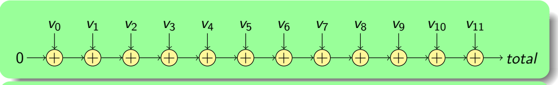
$$total = 0 \circledcirc vss$$

Transformation rules



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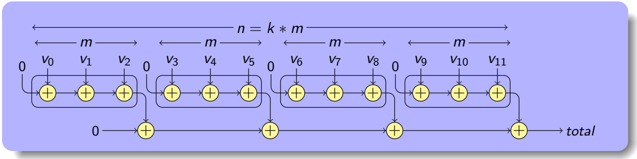

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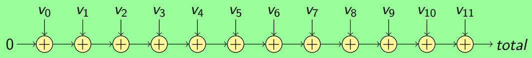

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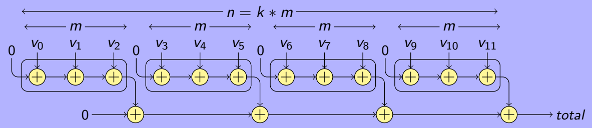
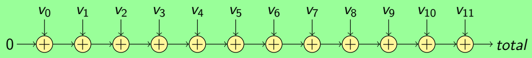


associative, neutral element

Transformation rules



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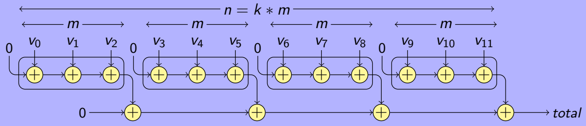
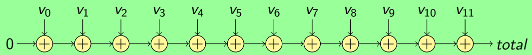


associative, neutral element

$$total = foldl\ (+)\ 0\ \$\ map\ (foldl\ (+)\ 0)\ vss$$

Transformation rules



$$total = foldl\ (+)\ 0\ vs$$

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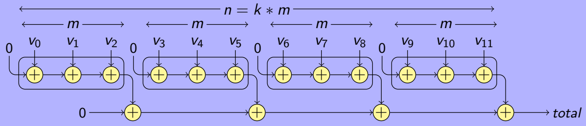


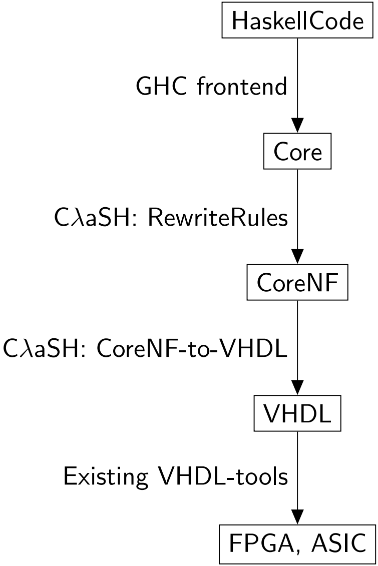
associative, neutral element

$$total = foldl\ (+)\ 0\ \$\ map\ (foldl\ (+)\ 0)\ vss$$

$$total = 0 \oplus (\widehat{0 \oplus vss})$$

The CλaSH Mechanism

CλaSH pipeline



Parsing
Desugaring
Type-checking

Remove polymorphism
Inlining
η-expansion
...

Translation into VHDL

Synthesis
Place-and-route

Rewrite system

Specification:

$$\begin{aligned} alu\ ADD &= (+) \\ alu\ MUL &= (*) \\ alu\ SUB &= (-) \end{aligned}$$

$$\textbf{data}\ \ OpCode = ADD \mid MUL \mid SUB$$

Note: $alu\ ADD :: Num\ a \Rightarrow a \to a \to a$

And: $alu\ ADD\ x\ y = x + y$

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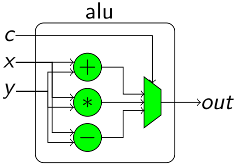

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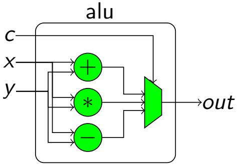

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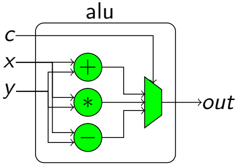
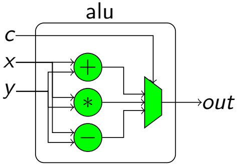


GHC ⇒ Core:

$$alu = \lambda c.\ \textbf{case}\ c\ \textbf{of} \quad \begin{aligned} &ADD \to (+) \\ &MUL \to (*) \\ &SUB \to (-) \end{aligned}$$

Rewrite system



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GHC $\Rightarrow$ Core:

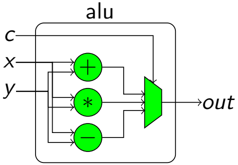
$$\mathit{alu} = \lambda c.\ \mathbf{case}\ c\ \mathbf{of} \begin{array}{l} \\ \mathit{ADD} \rightarrow (+) \\ \mathit{MUL} \rightarrow (*) \\ \mathit{SUB} \rightarrow (-) \end{array}$$

$\eta-$expansion:

$$\mathit{alu} = \lambda c.\ \lambda x.\ \lambda y. \left(\begin{array}{l} \mathbf{case}\ c\ \mathbf{of} \\ \quad \mathit{ADD} \rightarrow (+) \\ \quad \mathit{MUL} \rightarrow (*) \\ \quad \mathit{SUB} \rightarrow (-) \end{array} \right) x\ y$$

Rewrite system

$$alu = \lambda c.\,\lambda x.\,\lambda y.\left(\begin{array}{l}\textbf{case}\ c\ \textbf{of}\\ \quad ADD \rightarrow (+)\\ \quad MUL \rightarrow (*)\\ \quad SUB \rightarrow (-)\end{array}\right)\ x\ y$$



Rewrite system

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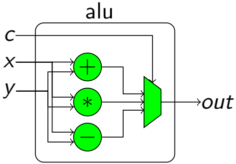
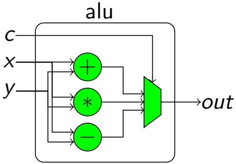


application propagation:

$$\begin{array}{rcl} alu & = & \lambda c\, x\, y.\ \textbf{case } c \textbf{ of} \\ & & \quad ADD \rightarrow (+)\ x\ y \\ & & \quad MUL \rightarrow (*)\ x\ y \\ & & \quad SUB \rightarrow (-)\ x\ y \end{array}$$

Rewrite system

$$alu \;=\; \lambda c.\,\lambda x.\,\lambda y.\left(\begin{array}{l}\textbf{case}\ c\ \textbf{of}\\ \quad ADD \rightarrow (+)\\ \quad MUL \rightarrow (*)\\ \quad SUB \rightarrow (-)\end{array}\right)\ x\ y$$



application propagation:

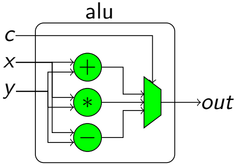
$$alu \;=\; \lambda c\,x\,y.\ \textbf{case}\ c\ \textbf{of}\quad \begin{array}{l} ADD \rightarrow (+)\ x\ y\\ MUL \rightarrow (*)\ x\ y\\ SUB \rightarrow (-)\ x\ y\end{array}$$

or, equivalently:

$$alu \;=\; \lambda c\,x\,y.\ \textbf{case}\ c\ \textbf{of}\quad \begin{array}{l} ADD \rightarrow x + y\\ MUL \rightarrow x * y\\ SUB \rightarrow x - y\end{array}$$

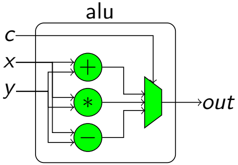
Rewrite system

$$\begin{aligned} alu \;=\; \lambda c\, x\, y.\ \textbf{case}\ c\ \textbf{of} & \\ & ADD \rightarrow x + y \\ & MUL \rightarrow x * y \\ & SUB \rightarrow x - y \end{aligned}$$



Rewrite system

$$alu \;=\; \lambda c\, x\, y.\ \textbf{case}\ c\ \textbf{of}\quad \begin{array}{l} ADD \to x + y \\ MUL \to x * y \\ SUB \to x - y \end{array}$$

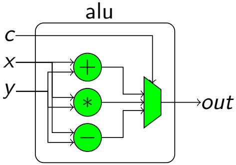


letification:

$$\begin{array}{lll} alu & = & \lambda c\, x\, y.\ \textbf{let} \\ & & \quad out = \textbf{case}\ c\ \textbf{of} \\ & & \qquad\qquad ADD \to x + y \\ & & \qquad\qquad MUL \to x * y \\ & & \qquad\qquad SUB \to x - y \\ & & \textbf{in} \\ & & \quad out \end{array}$$

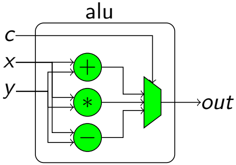
Rewrite system

$$
\begin{array}{lcl}
alu & = & \lambda c\,x\,y.\ \mathbf{let}\ out = \mathbf{case}\ c\ \mathbf{of} \\
 & & \qquad\qquad ADD \rightarrow x + y \\
 & & \qquad\qquad MUL \rightarrow x * y \\
 & & \qquad\qquad SUB \rightarrow x - y \\
 & & \quad\ \mathbf{in}\ out
\end{array}
$$



Rewrite system

$$alu \;=\; \lambda c\, x\, y.\ \textbf{let}\ out = \textbf{case}\ c\ \textbf{of} \\ \qquad ADD \to x + y \\ \qquad MUL \to x * y \\ \qquad SUB \to x - y \\ \textbf{in}\ out$$



subexpression extraction:

$$alu \;=\; \lambda c\, x\, y.\ \textbf{let}\ p = x + y \\ q = x * y \\ r = x - y \\ out = \textbf{case}\ c\ \textbf{of} \\ \qquad ADD \to p \\ \qquad MUL \to q \\ \qquad SUB \to r \\ \textbf{in}\ out$$

Thank you

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