5HC99 Embedded Vision Control

Feedback Control Systems

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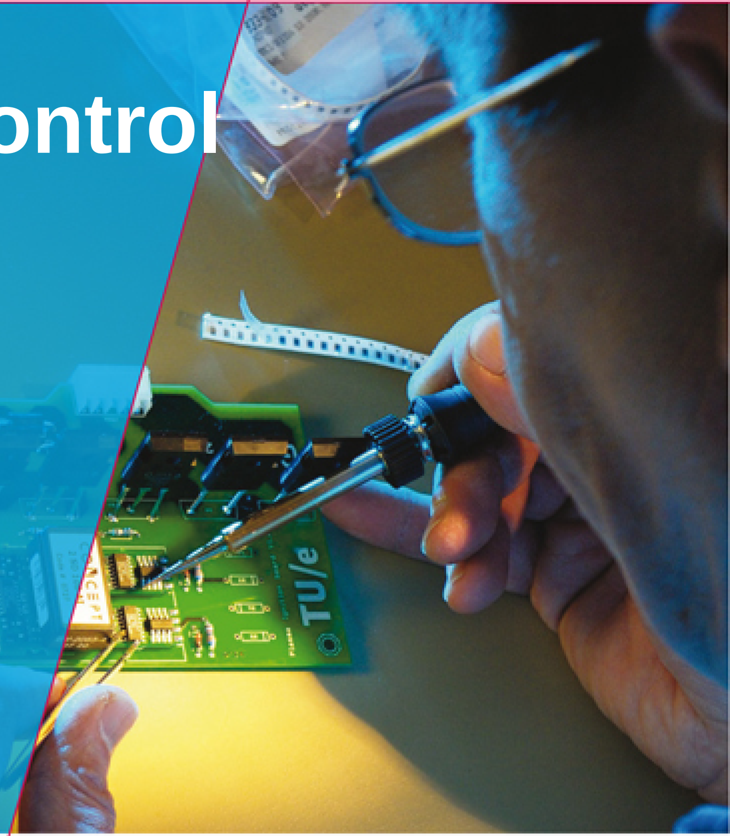

Technische Universiteit Eindhoven
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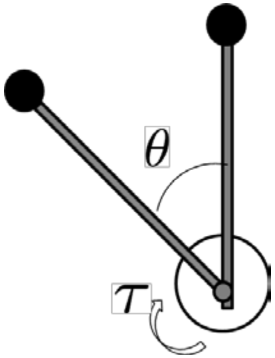
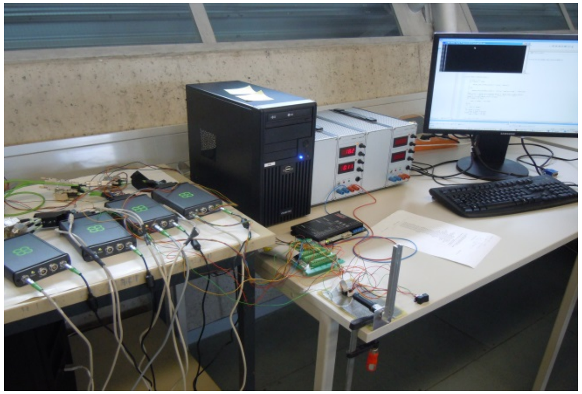
Where innovation starts

Example

Feedback control system: regulates the behavior of dynamical systems



Control objective: Keep the pendulum upright.



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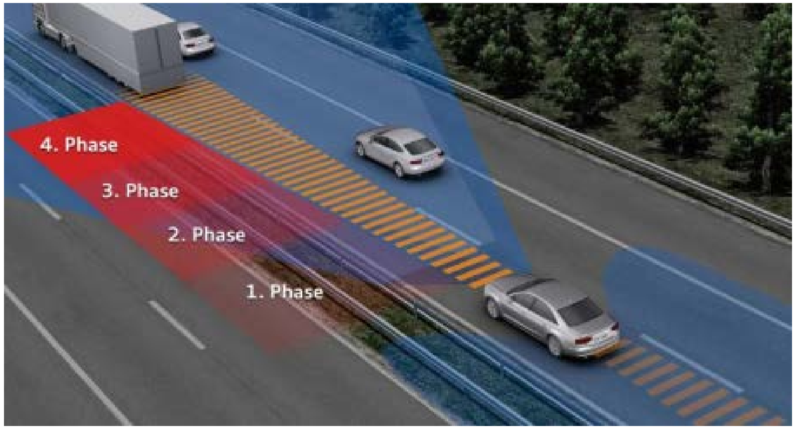
Feedback control applications



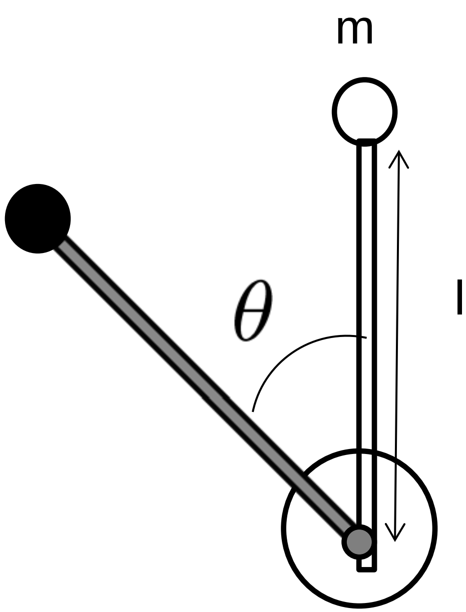
Adaptive cruise control

Pedestrian detection system

Suspension system

Modeling dynamical systems: system dynamics

- The system states changes with time...
- Generally, the dynamical systems are modeled by a set of differential equations...



$$\Rightarrow \quad ml^2\,\ddot{\theta} = \mathrm{mg\ l\ sin}\theta$$

4 Stability and basic principle

- Autonomous dynamical system: $\dot{x}$=ax
- Solution: x(t) = e^{at}x(0)
- Stability: a<0 implies x(t)→ 0 as t→ ∞
- Instability: a>0 implies x(t)→ ∞ as t→ ∞

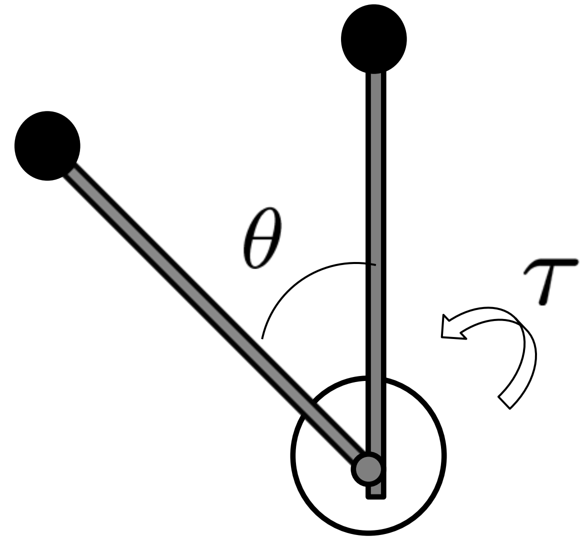
- General dynamical systems: $\dot{x}$=ax + u
- u = 0 (i.e., no control input): same as an autonomous system (open-loop system)
- Feedback controller: u = -Kx
- Closed-loop system: $\dot{x}$=ax + u = ax – Kx = (a-K)x
- If (a-K)<0, we have a stable closed-loop system
- Controller design: choose K such that (a-K)<0

- General dynamical system (state-space model):

$$\dot{x} = A\,x + Bu$$
$$y = Cx$$

Example: system dynamics

Given physical system: DC motor with inverted pendulum

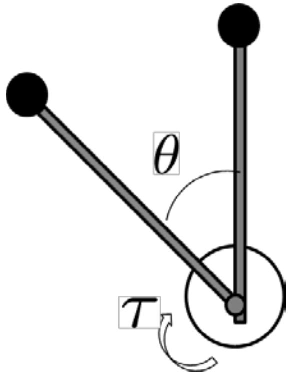


θ =shaft angular position

τ =applied motor torque

System dynamics: $\ddot{\theta} = 37\theta + 7.5\dot{\theta} + 6450\tau$

System dynamics→state-space model



(1) System dynamics: $\ddot{\theta} = 37\theta + 7.5\dot{\theta} + 6450\tau$

(2) Input and output:

$$y = \theta = x_1 \text{ (position)}$$
$$u = \tau \text{ (input motor torques)}$$

(3) States:

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

(4) State-space:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 37x_1 + 7.5x_1 + 6450u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 37 & 7.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 6450 \end{bmatrix} \mathrm{u}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

⇨

$$\dot{x} = A\,x + B\mathrm{u}$$
$$y = Cx$$

System dynamics→state-space model

(1) Double integrator: $\ddot{x}(t) = u(t)$

(2) Input and output:

$$\text{output} = y = x_1$$
$$\text{input} = u(t)$$

(3) States:

$$x_1 = x$$
$$x_2 = \dot{x}$$

(4) State-space:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$\Rightarrow$

$$\dot{x} = A\,x + B\mathrm{u}$$
$$y = Cx$$

Systems poles

- System model:

$$\dot{x} = A\,x + B\mathrm{u}$$
$$y = Cx$$

- System poles are the eigenvalues of A

- Double integrator:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

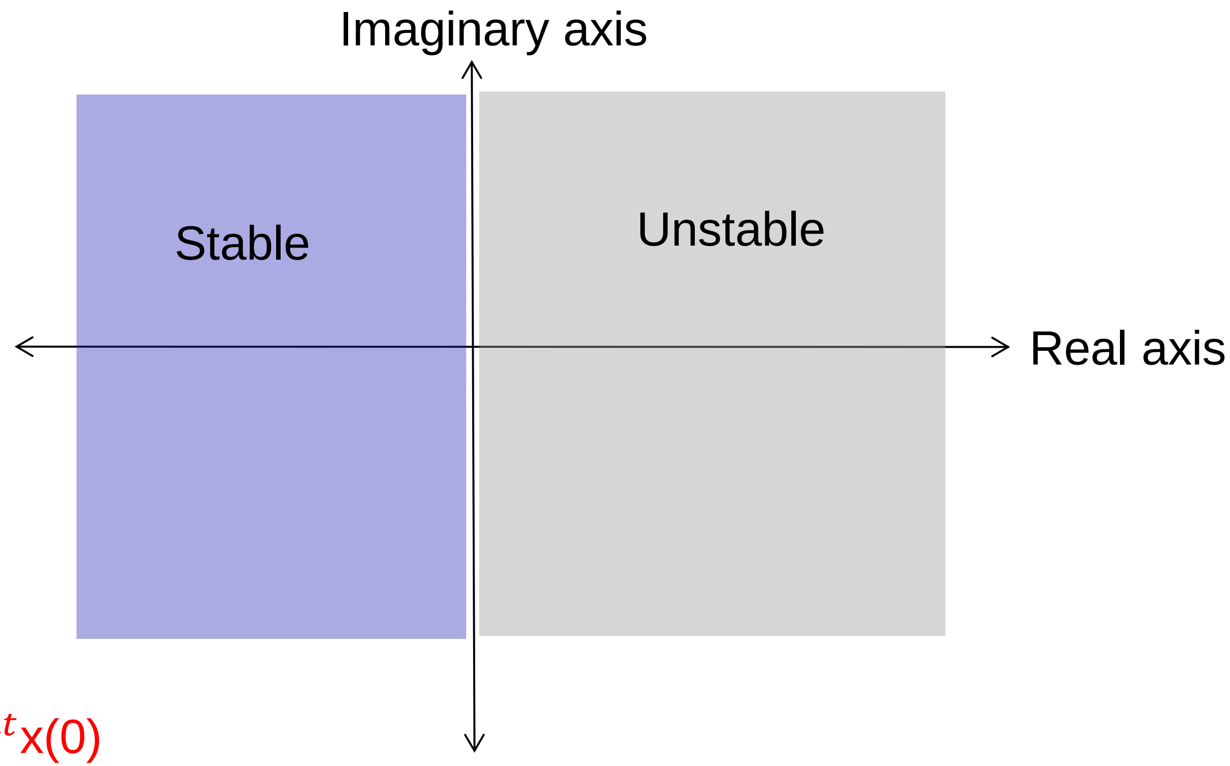
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}$$
$$determinant(\lambda I - A) = \lambda^2$$

$\Rightarrow$ Poles at 0, 0

Recall:
System: $\dot{x}$=ax
Solution: x(t) = e^{at}x(0)

Stability condition: continuous-time case

- Stable system
  - ⇒ All poles should have negative real part Ex. -3, (-3 + 2i)
- Marginally stable system
  - ⇒ One or multiple poles are on imaginary axis and all other poles have negative real parts Ex. 2i,
- Unstable system
  - ⇒ One or more poles with positive real part. Ex. +3, (+3 + 2i)



Recall:
System: $\dot{x}$=ax
Solution: x(t) = e^{at}x(0)

Simplest design problem

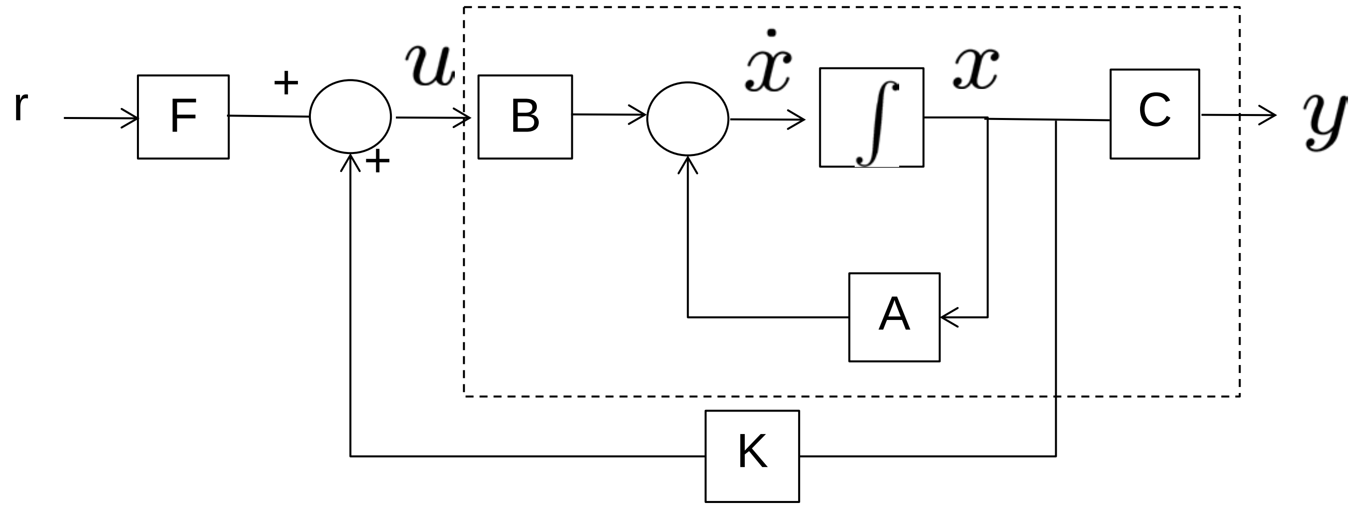
- We have a linear system given by state-space model –

$$\dot{x} = A\ x\ + Bu$$
$$y = Cx$$

- Objective – $y \rightarrow r$ as $time \rightarrow \infty$

- u = ?

State-feedback



$\dot{x} = Ax + Bu$
$y = Cx$

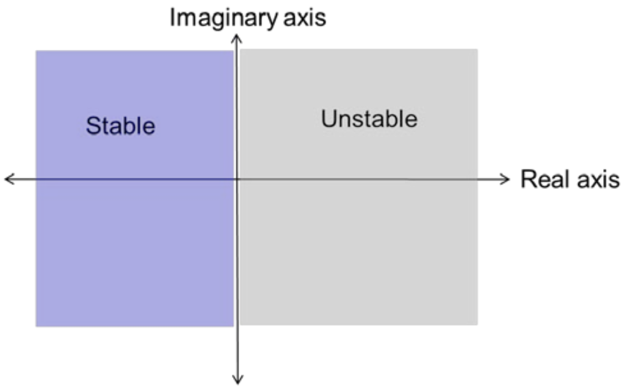
Open-loop system, i.e., with u=0 ⇨ $\dot{x} = Ax$

Closed-loop system with state-feedback control $u = Kx + Fr$,

⇨
$$\dot{x} = (A + BK)x + BFr$$
$$y = Cx$$

r = reference
K = feedback gain
F = static feedforward gain

Open-loop stability



$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\Downarrow \; u = 0$$

```
%% System matrix
A = [0   1
     5   6];
%% System open-loop poles
eigs(A)
```

Open-loop poles

```
>> 6.74
  -0.74
```

Unstable!

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{u}$$

Open-loop poles are at +6.74 and -0.74.

Assuming u=Kx, compute feedback gain K such that the closed-loop poles at -1 and -2.

$$u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ 5 + k_1 & 6 + k_2 \end{bmatrix} x$$

Example 2 -- contd

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5+k_1 & 6+k_2 \end{bmatrix} x$$

$$\Longrightarrow \quad determinant(\lambda I - \begin{bmatrix} 0 & 1 \\ 5+k_1 & 6+k_2 \end{bmatrix}) = 0$$

$$\Longrightarrow \quad \lambda^2 - (6+k_2)\lambda - (5+k_1) = 0$$

Desired pole locations at -1 and -2, that is the desired characteristics equation,

$$(\lambda+1)(\lambda+2) = 0$$

$$\Longrightarrow \lambda^2 + 3\lambda + 2 = 0$$

Comparing above two equations,

$$-6 - k_2 = 3 \rightarrow k_2 = -9,$$

$$\text{and } -5 - k_1 = 2 \rightarrow k_1 = -7, \quad \Longrightarrow \quad u = \begin{bmatrix} -7 & -9 \end{bmatrix} x$$

```
%% System matrix
A = [0   1
     5   6];
B = [0; 1];

%% Feedback gain
K = [-7 -9];
%% System closed-loop poles
eigs(A+B*K)
```

⇩

closed-loop poles

```
>> -1
  -2
```

⇩

Stable!

Systematic design: Ackermann's formula

- Choose the desired closed-loop poles are at

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix}$$

- Ackermann's formula:

$$K = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \gamma^{-1} H(A)$$

where

$$\gamma = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$H(A) = (A - \alpha_1 I)(A - \alpha_2 I)(A - \alpha_3 I) \cdots (A - \alpha_n I)$$

- Poles of (A+BK) are at $\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix}$

- Example 3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Open-loop poles are at +6.74 and -0.74. Compute feedback gain K such that the closed-loop poles at -1 and -2.

Ackermann's formula: $u = Kx$ and $K = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \gamma^{-1} H(A)$

$$\Longrightarrow \quad \gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix}$$

$$\Longrightarrow \quad \gamma^{-1} = \begin{bmatrix} -6 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 3 contd

$$\alpha_1 = -1, \alpha_2 = -2$$

$$\Rightarrow \quad H(A) = (A - \alpha_1 I)(A - \alpha_2 I) = (A + I)(A + 2I)$$

$$\Rightarrow \quad H(A) = A^2 + 3A + 2I$$

$$\Rightarrow \quad H(A) = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix}^2 + 3\begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad H(A) = \begin{bmatrix} 5 & 6 \\ 30 & 41 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad H(A) = \begin{bmatrix} 7 & 9 \\ 45 & 61 \end{bmatrix}$$

Example 3 contd

Feedback gain: $K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \gamma^{-1} H(A)$

$$\rightarrow K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ 47 & 61 \end{bmatrix}$$

$$\rightarrow K = \begin{bmatrix} -7 & -9 \end{bmatrix}$$

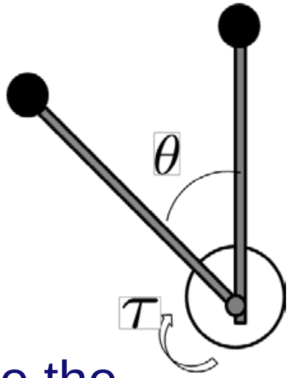
Cross-check

```
%% System matrix
A = [0   1
     5   6];
B = [0; 1];

gamma = [B A*B];
H = A^2 + 3*A+ 2*eye(2);

%% Feedback gain
K = -[0 1]*inv(gamma) *H
%% System open-loop poles
eigs(A+B*K)

%% Matlab function call
 K = -acker(A,B,[-1 -2])
%% System open-loop poles
 eigs(A+B*K)
```



- <u>Example</u>

  Given system dynamics $\ddot{\theta} = 37\theta + 7.5\dot{\theta} + 6450\tau$

  With input $\tau = K \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, find the feedback gain K to place the closed loop poles are located at -1 and -1.

$$\ddot{\theta} = 37\theta + 7.5\dot{\theta} + 6450\tau \quad \Rightarrow \quad x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}, \; u = \tau$$

$$\Rightarrow \quad \begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 37 & 7.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 6450 \end{bmatrix} \mathrm{u} \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

$$\gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 6450 \\ 6450 & 48375 \end{bmatrix} \quad \Rightarrow \quad \gamma^{-1} = \begin{bmatrix} -0.0012 & 0.0002 \\ 0.0002 & 0 \end{bmatrix}$$

$$H(A) = (A+I)(A+I) = (A^2 + 2A + I)$$

$$\rightarrow H(A) = \begin{bmatrix} 38 & 9.5 \\ 351.5 & 109.25 \end{bmatrix}$$

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \gamma^{-1} H(A) = \begin{bmatrix} -0.0059 & -0.0015 \end{bmatrix}$$

<u>Closed-loop poles</u>

Roots of $determinant(\lambda I - (A + BK))$

$\rightarrow -1$ and -1

Verify the results using MATLAB

- Example

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Is it possible to place the system poles?

$$K = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \gamma^{-1} H(A)$$

$$\gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$ ⇨ γ is not invertible ⇨ Pole placement not possible

⇨ The system not controllable!

Controllability

$$\dot{x} = A\ x + B\mathsf{u}$$
$$y = Cx$$

$$K = -\left[\begin{array}{cccc} 0 & 0 & \cdots & 1 \end{array}\right]\gamma^{-1}H(A)$$

- Check if you can control (i.e., place pole) the system ?

  Controllability matrix: $\gamma = \left[\begin{array}{ccccc} B & AB & A^2B & \cdots & A^{n-1}B \end{array}\right]$

  Controllability matrix should be invertible.

Feedforward gain

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Open-loop poles are at +6.74 and -0.74. Compute feedback gain K such that the closed-loop poles at -1 and -2.

We got: u = Kx = [-7 -9]x

x(t) = ? as t $\rightarrow \infty$

Recall:
System: $\dot{x}$=ax
Solution: x(t) = e^{at}x(0)

What if we want:
y(t) $\rightarrow r$ as $t \rightarrow \infty$

We apply the above method to achieve:
y(t)-r $\rightarrow 0$ as $t \rightarrow \infty$

Static feedforward gain

$$u = Kx + Fr$$

$K \rightarrow$pole placement

$F \rightarrow$ static feedforward gain

Closed-loop system

$$\dot{x} = (A + BK)x + BFr$$
$$y = Cx$$

Taking Laplace transform

$$\rightarrow X(s) = (sI - A - BK)^{-1}BFR(S)$$
$$\rightarrow Y(s) = CX(s) = C(sI - A - BK)^{-1}BFR(S)$$
$$\rightarrow G_{cl}(s) = \frac{Y(s)}{R(s)} = C(sI - A - BK)^{-1}BF$$

F should be chosen such that y(t) $\rightarrow$ r (constant) as t $\rightarrow \infty$ i.e.,

Using final value theorem $\Longrightarrow$ $\lim_{s\rightarrow 0} sY(s) = r$

$$\Longrightarrow \quad F = \frac{1}{C(-A-BK)^{-1}B}$$

Derivation of feedforward gain

- $\dot{x} = (A + BK)x + BFr$
- Taking Laplace transform: $sX(s) = (A + BK)X(s) + BF\,R(s)$
- Solve X(s): $X(s) = (sI - A - BK)^{-1}BFR(s)$
- R(s)=$\frac{r}{s}$
- From y=Cx, we get
$$Y(s) = CX(s) = C(sI - A - BK)^{-1}BFR(s)$$
- Final value theorem: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = r$ (this is what we want)
- The above implies using the expression of Y(s) and R(s):
$C(sI - A - BK)^{-1}BFr = r$
- Further, the above results in a following solution for F:

$$F = \frac{1}{C(-A-BK)^{-1}B}$$

Overall design

- Given system: $\dot{x} = Ax + B\text{u}$, $y = Cx$
- Control law: $u = Kx + Fr$

Objectives
(i) Place system poles
(ii) Achieve y $\rightarrow$ r as t $\rightarrow \infty$
(iii) Design K and F

$\Downarrow$

1. Check controllability of (A,B) $\rightarrow$ must be controllable. γ must be invertible.

$$\gamma = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

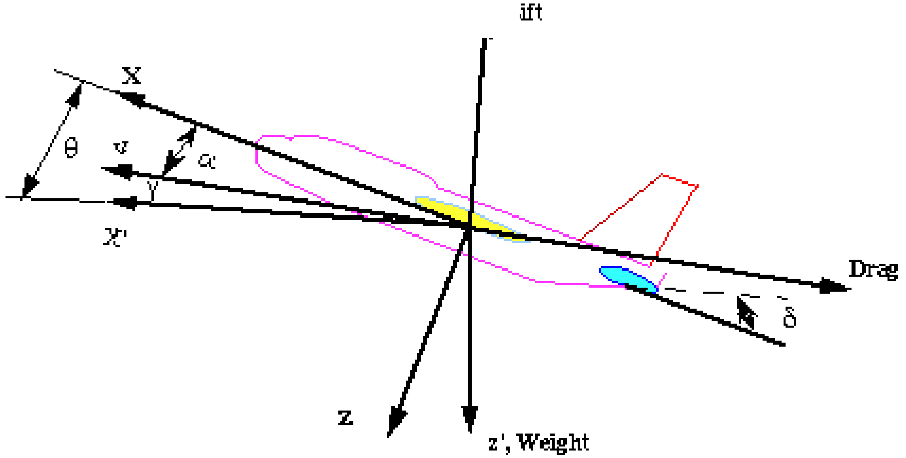
2. Apply Ackermann's formula $K = -\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \gamma^{-1} H(A)$

3. Feedforward gain $F = \dfrac{1}{C(-A-BK)^{-1}B}$

Design example

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \delta$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$



Design a controller such that pitch angle $\theta = 0.03$ rad with time.

Source: /www.library.cmu.edu/ctms/

- Choose closed-poles at stable locations

$$\alpha = \begin{bmatrix} -4 & -3 & -3 \end{bmatrix}$$

- Controllability matrix $\gamma = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$

$$det(\gamma) = -0.0044$$

- Feedback gain $K = \begin{bmatrix} 167.1 & 2366.2 & -202.9 \end{bmatrix}$

- Feedforward gain $\mathrm{F} = 220.90$

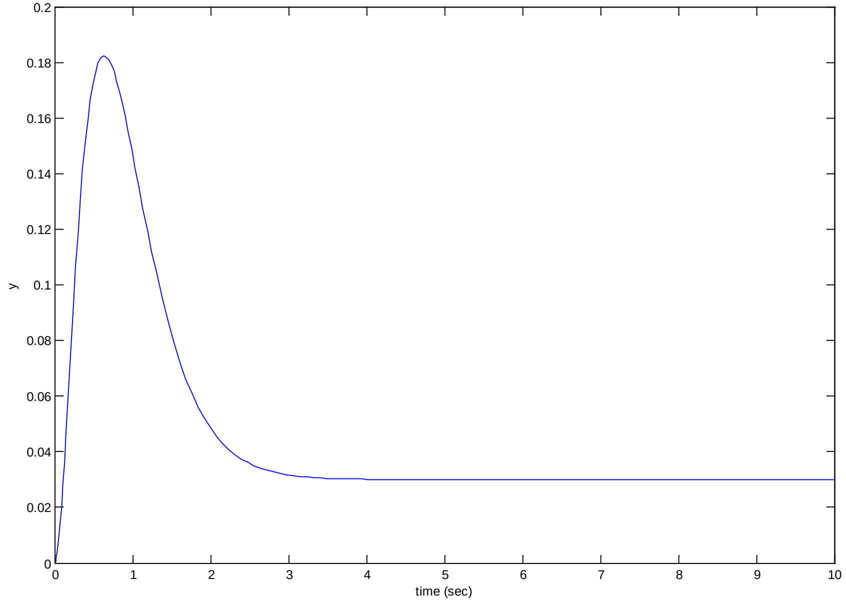
- $\delta = \mathrm{K}\begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \mathrm{F} \times 0.03$

$$\begin{bmatrix}\dot{\alpha}\\ \dot{q}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}-0.313 & 56.7 & 0\\ -0.0139 & -0.426 & 0\\ 0 & 56.7 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ q\\ \theta\end{bmatrix} + \begin{bmatrix}0.232\\ 0.0203\\ 0\end{bmatrix}\delta$$

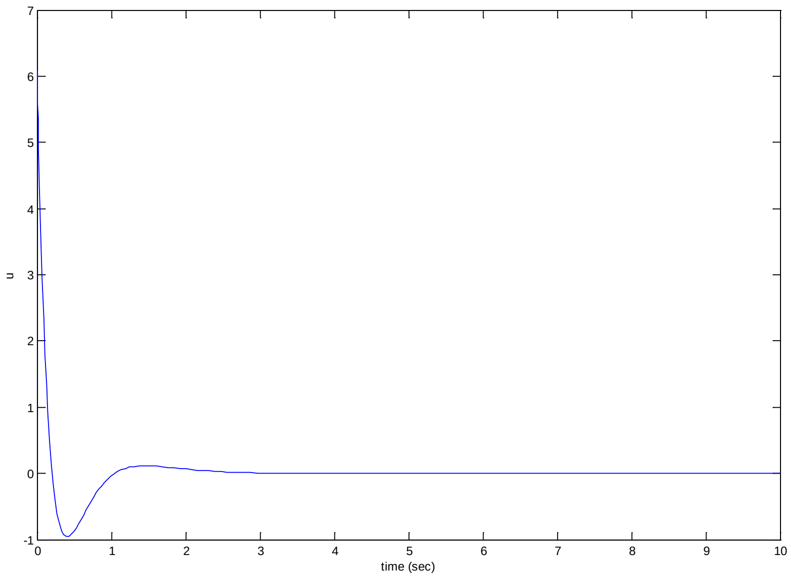
$$y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ q\\ \theta\end{bmatrix}$$

$$\delta = \mathrm{K}\begin{bmatrix}\alpha\\ q\\ \theta\end{bmatrix} + \mathrm{F} \times 0.03$$

Output θ

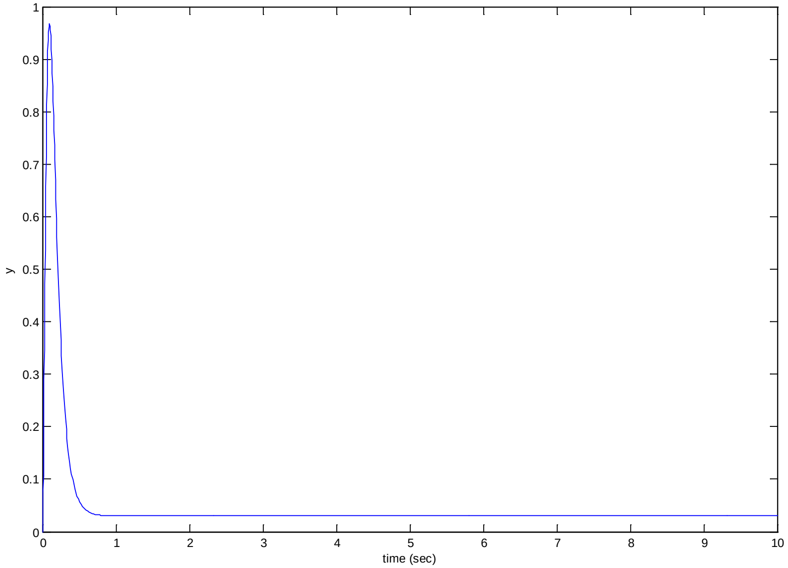


Input δ

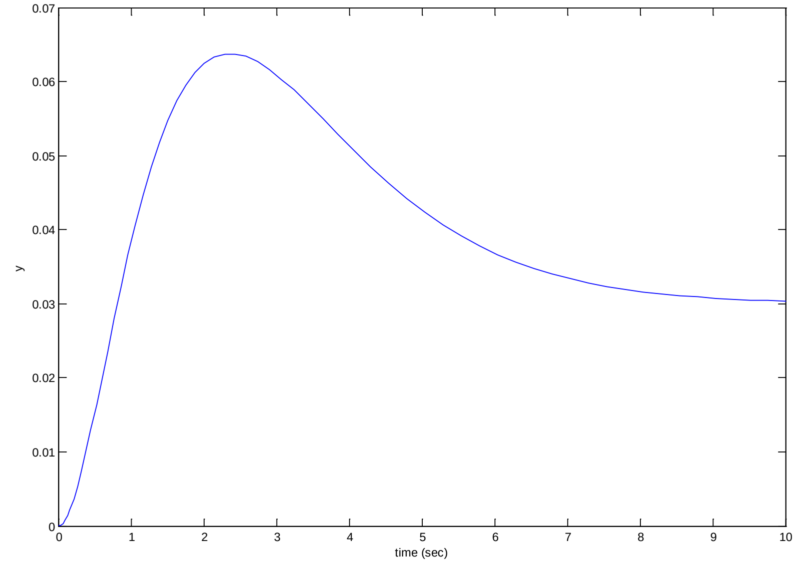


Design consideration: response time

$$\alpha = \begin{bmatrix} -10 & -30 & -30 \end{bmatrix}$$



$$\alpha = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$$



Aggressive poles $\rightarrow$ faster response

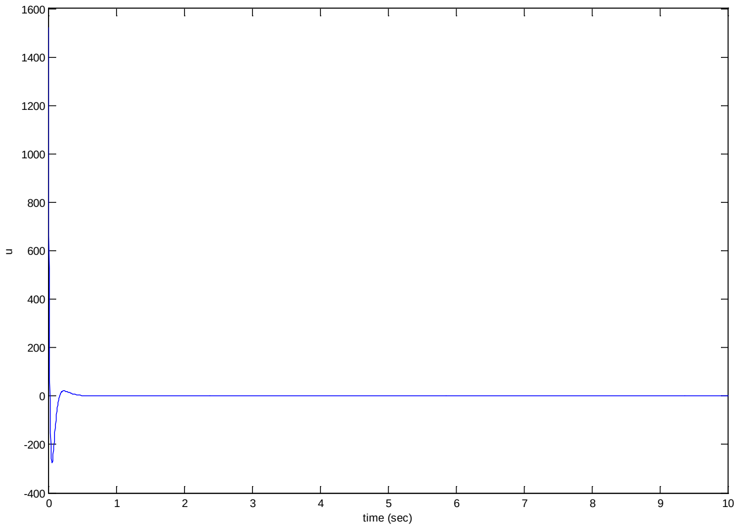
Recall:

System: $\dot{x}$=ax

Solution: x(t) = e^{at}x(0)

Design consideration: input saturation

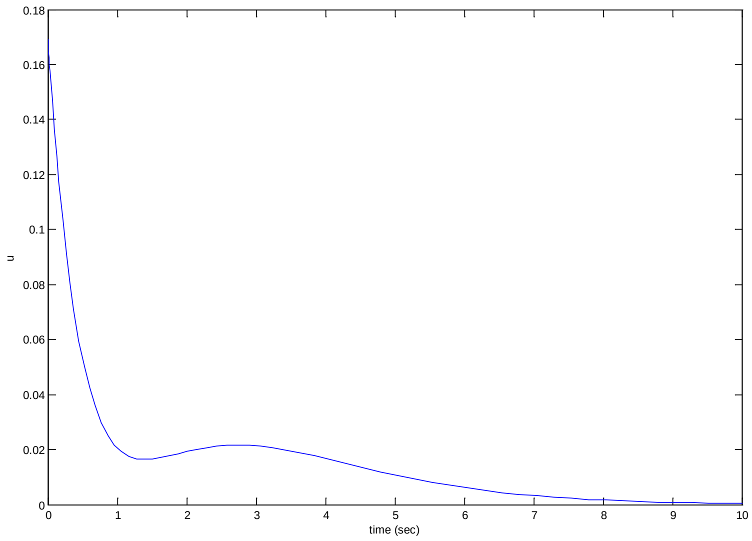
$$\alpha = \begin{bmatrix} -10 & -30 & -30 \end{bmatrix}$$



$$\Downarrow$$

$$max(u) = 1522$$

$$\alpha = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$$



$$\Downarrow$$

$$max(u) = 0.1691$$

Aggressive poles $\rightarrow$ higher input signal

Pole choice

- Reasonable to represent system performance by the poles

<u>Time- domain</u>
settling time, rise time, overshoot

<u>Frequency-domain</u>
Gain Margin,
Phase Margin

<u>External constraints</u>
Input saturation

→ Specification → Desired poles

Apply the method
to design a controller
to achieve the desired poles